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Dirac Canonical Quantization of Composite Fermions QED

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Abstract Composite Fermions QED is quantized by using the Dirac's canonical formalism for constrained systems. As a strategy, we first work out the constraints (including primary and secondary constraints), combine two first-class constraints, introduce Coulomb gauge and its stationary as gauge conditions, and then quantize, replacing the Dirac brackets with quantum commutators.

Keywords Dirac canonical quantization · Chern-Simons · Gauge condition · Dirac conjecture

In this letter we investigate composite fermions QED (quantum electrodynamics) [1–5] with Coulomb gauge and its stationary in Dirac's canonical formalism for constrained systems [6, 7]. In order to illustrate FQHE (fractional quantum Hall effect) phenomenon with the states higher than the LLL (least Landau level), Jian supplied a new path that is composite fermion [8–10]. Another way is quantum field theory, in which abelian Chern-Simons fields are introduced in the Lagrangian of the constrained systems discussed. Chern-Simons gauge fields can change the statistical properties of particles (while Bosons or Fermions) by adding quantum flux to them. Though the composite particle concept is familiar to theoretical physicists, Dirac's canonical formalism of composite Boson systems and composite Fermion systems are rarely discussed. In this letter we quantize composite Fermion QED in Dirac's canonical formalism, but its quantum symmetries will be discussed in another

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paper [11]. To quantize this system, we select Coulomb gauge and its stationary as gauge conditions with respect to first-class constraints in Dirac's sense. The selection of gauge conditions need to be discussed deeply. Absolutely, when all the first-class constraints in constrained canonical systems are functions of abelian gauge fields A_μ and corresponding momenta π_μ, Coulomb gauge is effective. This topic will be discussed deeply and widely [12].

We consider composite Fermions QED, selecting $c=1$, which has Lagrangian density as

$$\mathscr{L}=\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{e\pi}{2\theta\Phi_0}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho-e\bar{\psi}\gamma^\mu\psi A_\mu-e\bar{\psi}\gamma^\mu\psi a_\mu, \quad (1)$$

where $\varepsilon^{\mu\nu\rho}$ is Levi-Civita symbol, an antisymmetric tensor, ψ is Dirac's spinor containing four complex parts, $\bar{\psi}=\psi^+\gamma^0$, γ^μ is Dirac matrix, e is the charge of electron, Φ_0 is the unit of magnetic flux, θ is gauge parameter of Chern-Simons fields, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ are called electromagnetic tensor, A_μ are electromagnetic fields, and a_μ are Chern-Simons gauge fields. In the right hand of (1), the first term denotes spinor fields, the second describes electromagnetic fields, the third are from gauge Chern-Simons fields, the rest two terms can be taken as the spinor fields coupling to electromagnetic fields and Chern-Simons fields, respectively. Abelian Chern-Simons gauge fields a_μ are the common electromagnetic fields with the relations $a_\mu=v_{\mathit{eff}}A_\mu=\frac{-2p}{2p\pm1}$ $(p=0,1,2,\ldots)$ [13–16]. Therefore, the Lagrangian density of (1) can be written as

$$\mathscr{L}=\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{e\pi v_{\mathit{eff}}^2}{2\theta\Phi_0}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho-(1+v_{\mathit{eff}})e\bar{\psi}\gamma^\mu\psi A_\mu. \quad (2)$$

Canonical momenta $\bar{\pi}_a,\pi_a,\pi_\mu$ conjugated to fields variables $\bar{\psi}$, ψ, A_μ are

$$\begin{aligned}&\bar{\pi}_a=\frac{\delta\mathscr{L}}{\delta\dot{\bar{\psi}}_a}=0; \qquad \pi_a=\frac{\delta\mathscr{L}}{\delta\dot{\psi}_a}=i\bar{\psi}\gamma^0;\\ &\pi_i=\frac{\delta\mathscr{L}}{\delta\dot{A}_i}=-F^{0i}+\frac{e\pi v_{\mathit{eff}}^2}{2\theta\Phi_0}\varepsilon^{ij}A_j; \qquad \pi_0=\frac{\delta\mathscr{L}}{\delta\dot{A}_0}=0;\end{aligned} \quad (3)$$

respectively, here $\varepsilon^{ij}=\varepsilon^{0ij}$. The canonical Hamiltonian density of the system is given by

$$\begin{aligned}\mathscr{H}_C&=\bar{\pi}_a\dot{\bar{\psi}}+\pi_a\dot{\psi}+\pi_\mu\dot{A}_\mu-\mathscr{L},\\ &=\frac{1}{2}\pi_i^2-\pi_i\partial_iA_0+\frac{1}{4}F_{ij}F^{ij}+\frac{1}{2}\left(\frac{e\pi v_{\mathit{eff}}^2}{2\theta\Phi_0}\varepsilon^{ij}A_j\right)^2,\\ &\quad-\bar{\psi}(i\gamma^i\partial_i-m)\psi+\bar{\psi}m\psi-\frac{e\pi v_{\mathit{eff}}^2}{2\theta\Phi_0}\varepsilon^{\mu\nu i}A_\mu\partial_\nu A_i+(1+v_{\mathit{eff}})e\bar{\psi}\gamma^\mu\psi A_\mu.\end{aligned} \quad (4)$$

The primary constraints are

$$\bar{\phi}_a^0=\bar{\pi}_a\approx 0, \quad (5)$$

$$\phi_a^0=\pi_a-i(\bar{\psi}\gamma^0)_a\approx 0, \quad (6)$$

$$\phi_0^0=\pi_0\approx 0, \quad (7)$$

respectively. Then the total Hamiltonian density for the system is

$$\begin{aligned}
\mathscr{H}_T &= \mathscr{H}_C + \bar{\lambda}_a \bar{\phi}_a^0 + \lambda_a \phi_a^0 + \lambda_0 \phi_0^0 \\
&= \bar{\pi}_a \dot{\bar{\psi}} + \pi_a \dot{\psi} + \pi_\mu \dot{A}_\mu - \mathscr{L} \\
&= \frac{1}{2}\pi_i^2 - \pi_i \partial_i A_0 + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2}\left(\frac{e\pi v_{eff}^2}{2\theta \Phi_0}\varepsilon^{ij} A_j\right)^2 \\
&\quad - \bar{\psi}(i\gamma^i \partial_i - m)\psi + \bar{\psi} m \psi - \frac{e\pi v_{eff}^2}{2\theta\Phi_0}\varepsilon^{\mu\nu i} A_\mu \partial_\nu A_i + (1 + v_{eff}) e \bar{\psi}\gamma^\mu \psi A_\mu \\
&\quad + \bar{\lambda}_a \bar{\phi}_a^0 + \lambda_a \phi_a^0 + \lambda_0 \phi_0^0,
\end{aligned} \tag{8}$$

where $\bar{\lambda}_a(x)$, $\lambda_a(x)$ $(a = 1, 2, 3, 4)$ and λ_0^0 are Lagrange multipliers with respect to $\bar{\phi}_a^0$, ϕ_a^0 $(i = 1, 2, 3)$ and ϕ_0^0. The total Hamiltonian is

$$H_T = \int_V d^3x \mathscr{H}_T. \tag{9}$$

The stationary of the primary constraint $\bar{\phi}_a^0$, $\{\bar{\phi}_a^0, H_T\} \approx 0$, leads to the equation for determining the multiplier λ_a

$$i\lambda_a \gamma^0 = (1 + v_{eff}) e \gamma^\mu \psi_a A_\mu - (i\gamma^i \partial_i - m)\psi_a + m\psi. \tag{10}$$

The stationary of the constraint ϕ_a^0, $\{\phi_a^0, H_T\} \approx 0$, gives

$$i\gamma^0 \bar{\lambda}_a = \bar{\psi}_a (i\gamma^i \partial_i - m) + \bar{\psi} m - (1 + v_{eff}) e \bar{\psi}_a \gamma^\mu A_\mu. \tag{11}$$

The stationary of the primary constraint ϕ_0^0, $\{\phi_0^0, H_T\} \approx 0$, yields the secondary constraint as

$$\phi_1^1 = -\pi_i \partial_i \delta(x - y) + \frac{e\pi v_{eff}^2}{2\theta\Phi_0}\varepsilon^{ji}\partial_i A^j + (1 + v_{eff}) e \bar{\psi}\gamma^0 \psi \approx 0. \tag{12}$$

The stationary of the secondary constraint ϕ_1^1, does not produce any new constraint.

Let us denote $\Lambda_1 = \pi_0 \approx 0$; one finds a linear combination of the constraints $\bar{\phi}_a^0$, ϕ_a^0 and ϕ_1^1 as

$$\begin{aligned}
\Lambda_2 &= \phi_1^1 + ie(1 + v_{eff})(\bar{\psi}_a \bar{\phi}_a^0 - \phi_a^0 \psi_a) \\
&= -\pi_i \partial_i \delta(x - y) + \frac{e\pi v_{eff}^2}{2\theta\Phi_0}\varepsilon^{ji}\partial_i A^j \\
&\quad + ie(1 + v_{eff})\bar{\psi}_a \bar{\pi}_a - ie(1 + v_{eff})\pi_a \psi_a \\
&\approx 0.
\end{aligned} \tag{13}$$

It is easy to check that

$$\{\Lambda_1, \Lambda_2\} \approx 0, \tag{14}$$

$$\{\Lambda_1, \bar{\phi}_a^0\} \approx 0, \tag{15}$$

$$\{\Lambda_1, \phi_a^0\} \approx 0, \tag{16}$$

$$\{\Lambda_2, \bar{\phi}_a^0\} = -ie(1+v_{eff})\phi_a^0\delta(x-y) \approx 0, \tag{17}$$

$$\{\Lambda_2, \phi_a^0\} = -ie(1+v_{eff})\bar{\phi}_a^0\delta(x-y) \approx 0, \tag{18}$$

$$\{\bar{\phi}_a^0, \phi_a^0\} = i\gamma^0\delta(x-y). \tag{19}$$

Hence, the constraints Λ_1 and Λ_2 are first class, while the constraints ϕ_a^0 and $\bar{\phi}_a^0$ are second class.

By denoting $\theta_i = (\bar{\psi}_a, \psi_a)$, we can obtain matrix elements by defining $\Delta_{ij} = \{\theta_i, \theta_j\}$ as

$$\Delta(x,y) = i\begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma_T^0 \end{pmatrix}\delta(x-y). \tag{20}$$

Its inverse matrix is

$$\Delta^{-1}(x,y) = -i\begin{pmatrix} \gamma_T^0 & 0 \\ 0 & \gamma^0 \end{pmatrix}\delta(x-y). \tag{21}$$

The Dirac's Bracket of the two variables $A(x)$ and $B(y)$ is

$$\begin{aligned}\{A(x), B(y)\}_{D(\Delta)} &= \{A(x), B(y)\} - i(\gamma^0)_{ab}\int d^3z\{A(x), \bar{\phi}_a^0(z)\}\{\phi_b^0(z), B(y)\} \\ &\quad - i(\gamma^0)_{ab}\int d^3z\{A(x), \phi_a^0(z)\}\{\bar{\phi}_b^0(z), B(y)\},\end{aligned} \tag{22}$$

where $\{A(x), B(y)\}$ is Bose-Fermi bracket, which is defined as

$$\begin{aligned}&\{B_1(x), B_2(x')\} \\ &= \int d^3y\left[\left(\frac{\partial B_1(x)\partial B_2(x')}{\partial\varphi^i(y)\partial\pi_i(y)} - B_1 \leftrightarrow B_2\right) - \left(\frac{\partial B_1(x)\partial B_2(x')}{\partial\psi^a(y)\partial\pi_a(y)} - B_1 \leftrightarrow B_2\right)\right],\end{aligned} \tag{23}$$

$$\begin{aligned}&\{F(x), B(x')\} \\ &= \int d^3y\left[\left(\frac{\partial F(x)\partial B(x')}{\partial\varphi^i(y)\partial\pi_i(y)} - F \leftrightarrow B\right) - \left(\frac{\partial F(x)\partial B(x')}{\partial\psi^a(y)\partial\pi_a(y)} + F \leftrightarrow B\right)\right],\end{aligned} \tag{24}$$

$$\begin{aligned}&\{F_1(x), F_2(x')\} \\ &= \int d^3y\left[\left(\frac{\partial F_1(x)\partial F_2(x')}{\partial\varphi^i(y)\partial\pi_i(y)} + F_1 \leftrightarrow F_2\right) - \left(\frac{\partial F_1(x)\partial F_2(x')}{\partial\psi^a(y)\partial\pi_a(y)} + F_1 \leftrightarrow F_2\right)\right],\end{aligned} \tag{25}$$

here $B_1(x)$, $B_2(x')$, $B(x)$ denote Bose fields, and $F(x)$, $F_1(x)$, $F_2(x')$ denote Fermi fields. Then Dirac's brackets corresponding to the second constraints of all canonical variables and canonical momenta are deduced as

$$\{A_\mu(x), \pi_\nu(y)\} = \delta_\mu^\nu\delta(x-y), \tag{26}$$

$$\{\bar{\psi}_a, \psi_b\}_{D(\Delta)} = i(\gamma^0)_{ab}\delta(x-y), \tag{27}$$

$$\{\bar{\psi}_a, \bar{\pi}_b\}_{D(\Delta)} = -(\gamma^0)_{ab}\gamma^0\delta(x-y), \tag{28}$$

and the rest are zeros.

In Dirac's canonical quantization, for each first class constraint, one must choose a gauge condition. Consider the Coulomb gauge and its stationary as

$$\Omega_1 = \partial_i \pi_i + \nabla^2 A_0 \approx 0, \tag{29}$$

$$\Omega_2 = \partial_i A_i \approx 0. \tag{30}$$

Denoting $\psi_i = (\Omega_1, \Omega_2, \Lambda_1, \Lambda_2)$, by defining the matrix element $G_{ij}(x,y) = \{\psi_i(x), \psi_j(y)\}$, one can calculate the following matrix as

$$G(x,y) = \begin{pmatrix} 0 & -\nabla^2 & \nabla^2 & 0 \\ \nabla^2 & 0 & 0 & -\nabla^2 \\ -\nabla^2 & 0 & 0 & 0 \\ 0 & \nabla^2 & 0 & 0 \end{pmatrix} \delta(x-y). \tag{31}$$

Its inverse matrix is

$$G^{-1}(x,y) = \frac{1}{4\pi|x-y|} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{32}$$

Basing on the matrix G^{-1}, one can give Dirac's brackets by

$$\begin{aligned} \{A(x), B(y)\}_D &= \{A(x), B(y)\}_{D(\Delta)} \\ &\quad - \int d^3z_1 d^3z_2 \{A(x), \psi_i(z_1)\}_{D(\Delta)} G_{ij}^{-1} \{\psi_j(z_2), B(y)\}_{D(\Delta)}. \end{aligned} \tag{33}$$

The Dirac's brackets of $A_i, \pi_i, \bar{\psi}_a, \bar{\pi}_a, \psi_a, \pi_a$ can be deduced as

$$\{A_i(x), \pi_j(y)\}_D = \delta_i^j \delta(x-y) - \partial_i \partial^j \frac{1}{2\pi|x-y|}, \tag{34}$$

$$\{\bar{\psi}_a, \bar{\pi}_a\}_D = i(\gamma^0)_{ab} \delta(x-y), \tag{35}$$

$$\{\psi_a, \pi_a\}_D = 0, \tag{36}$$

$$\{\bar{\psi}_a, \psi_a\}_D = -(\gamma^0)_{ab} \gamma^0 \delta(x-y). \tag{37}$$

Based on the relation between Dirac's bracket and the quantum theory is

$$\{\varphi_i, \varphi_j\}_D = -i[\varphi_i, \varphi_j]_- \tag{38}$$

for Bose fields, and

$$\{\psi_\alpha, \bar{\psi}_\beta\}_D = -i[\psi_\alpha, \bar{\psi}_\beta]_+ \tag{39}$$

for Fermi fields, the quantum commute relations of fundamental canonical fields variables and momenta $A_i, \pi_i, \bar{\psi}_a, \psi_a, \bar{\pi}_a$ and π_a are given by

$$[A_i(x), \pi_j(y)]_- = -i\delta_i^j \delta(x-y) + i\partial_i \partial^j \frac{1}{2\pi|x-y|}, \tag{40}$$

$$[\bar{\psi}_a, \bar{\pi}_a]_+ = (\gamma^0)_{ab} \delta(x-y), \tag{41}$$

$$[\psi_a, \pi_a]_+ = 0, \tag{42}$$

$$[\bar{\psi}_a, \psi_a]_+ = i(\gamma^0)_{ab}\gamma^0\delta(x-y). \tag{43}$$

The quantum symmetries of this constrained system are worthwhile to be discussed further in path-integral quantization formalism. That can show how the composite Fermions interact.

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